

# FEKETE'S SUBADDITIVE LEMMA REVISITED.

LÁSZLÓ TAPOLCAI GREINER

ABSTRACT. The subject of this paper is an extension of the Fekete's Subadditive Lemma for a set of submultiplicative functionals on infinite product of compact spaces. Our method it can be considered as an unfolding of the ideas [1] Theorem 3.1 and our main result is the extension of the symbolic dynamics results of [4].

## 1. EXTREMAL SUBMULTIPLICATIVE SEQUENCES

**1.1. Subadditive Lemma.** To begin with we recall the Subadditive Lemma [1]. If the sequence  $\{a_n \mid n = 1, 2, \dots\}$  of real numbers is subadditive in the sense

$$a_{n+m} \leq a_n + a_m \quad n, m = 1, 2, \dots,$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} a_n = \inf_n \frac{1}{n} a_n$$

where  $\inf_n$  means infimum.

We shall use also a *multiplicative form of this lemma*.

**Lemma 1.1.** *Let the sequence  $\{c_n \mid n = 1, 2, \dots\}$  of positive numbers be submultiplicative in the sense*

$$c_{n+m} \leq c_n c_m \quad n, m = 1, 2, \dots,$$

then

$$\lim_{n \rightarrow \infty} c_n^{\frac{1}{n}} = \inf_n c_n^{\frac{1}{n}}$$

where  $\inf_n$  means infimum.

*Proof.* the proof follows from the obvious connection that  $a_n = \log c_n$  is subadditive if  $c_n$  is submultiplicative.  $\square$

Let  $\{X_i \mid i = 1, 2, \dots\}$  be a sequence of compact spaces and consider the product space  $\prod_{i=1}^{\infty} X_i$ . We shall call *flow* the elements of  $\prod_{i=1}^{\infty} X_i$  and *word* the elements of a finite product of  $\{X_i \mid i = 1, 2, \dots\}$ .

More particularly, if  $\sigma_i$  is an element of  $X_i$ , then  $\sigma_i \sigma_{i+1} \dots \sigma_j$  is a word in  $\prod_{n=i}^j X_n$  and  $\sigma = \sigma_1 \sigma_2 \dots$  is a flow.

Let  $\phi$  be a positive functional on the words, which is *submultiplicative* in the following sense. If  $\sigma_i \sigma_{i+1} \dots \sigma_j$  is a word and  $i < k < j$ , then

$$\phi(\sigma_i \sigma_{i+1} \dots \sigma_j) \leq \phi(\sigma_i \sigma_{i+1} \dots \sigma_k) \phi(\sigma_{k+1} \sigma_{k+2} \dots \sigma_j).$$

---

2010 *Mathematics Subject Classification.* Primary 39B12; Secondary 68Q99.

*Key words and phrases.* submultiplicative functional, joint contraction, Turing machine, entropy.

If  $\phi$  is continuous on each  $\{X_i \mid i = 1, 2, \dots\}$  then there is  $\sigma_1^{[n]} \dots \sigma_n^{[n]}$  for every  $n$  that

$$\phi(\sigma_1^{[n]} \dots \sigma_n^{[n]}) = \sup\{\phi(\sigma_1 \dots \sigma_n) : \sigma \in \Pi_{i=1}^\infty X_i\}$$

since  $\{X_i \mid i = 1, 2, \dots\}$  are compact sets. If

$$(1.1) \quad \Phi_n = \phi(\sigma_1^{[n]} \dots \sigma_n^{[n]})$$

then it is easy to verify that the sequence  $\{\Phi_n \mid n = 1, 2, \dots\}$  of positive numbers is submultiplicative. Hence, it follows from the Submultiplicative Lemma, that  $\{\Phi_n^{\frac{1}{n}} \mid n = 1, 2, \dots\}$  is convergent and the limit is the *infimum* of  $\{\Phi_n^{\frac{1}{n}} \mid n = 1, 2, \dots\}$ .

Notice, that the words  $\{\sigma_1^{[n]} \dots \sigma_n^{[n]} \mid n = 1, 2, \dots\}$ , where  $\phi$  reach its maximum in  $\{X_n \mid n = 1, 2, \dots\}$ , are not the prefix of a single flow  $\sigma = \sigma_1 \sigma_2 \dots$  ( $\sigma_i \in X_i$ ).

Let  $\Phi_*$  be the limit of  $\{\Phi_n^{\frac{1}{n}} \mid n = 1, 2, \dots\}$ . For every flow  $\sigma \in \Pi_{i=1}^\infty X_i$ , there is also a limit

$$\phi_*(\sigma) = \lim_{n \rightarrow \infty} \phi(\sigma_1 \dots \sigma_n)^{\frac{1}{n}}$$

obviously,  $\phi_*(\sigma) \leq \Phi_*$ .

Our main result is that there exists  $\sigma$  with  $\phi_*(\sigma) = \Phi_*$ .

**Theorem 1.2.** *There is  $\sigma \in \Pi_{i=1}^\infty X_i$  with*

$$(1.2) \quad \phi(\sigma_1 \sigma_2 \dots \sigma_n) \geq \Phi_*^n$$

for  $n = 1, 2, \dots$  i.e. (1.2) holds for every prefix of  $\sigma$ .

*Proof.* Consider the nested sequence

$$(1.3) \quad \mathcal{T}_n = \{\sigma_1, \sigma_2 \dots \sigma_n : \phi(\sigma_1, \sigma_2 \dots \sigma_k) \geq \Phi_*^k \mid k = 1, 2, \dots, n\}$$

of compact sets. If  $\mathcal{T}_n \neq \emptyset$  for every  $n$ , then by the finite intersection property,

$$\bigcap_{n=1}^\infty \mathcal{T}_n \neq \emptyset$$

and each  $\sigma$  in  $\bigcap_{n=1}^\infty \{\mathcal{T}_n \mid n = 1, 2, \dots\}$  satisfies (1.2) for  $n = 1, 2, \dots$ .

Hence, if we prove that each set  $\mathcal{T}_n$  is not empty then we have done.

Suppose that there is  $n$  that  $\mathcal{T}_n = \emptyset$  and we get to a contradiction. If there is  $n$  that  $\mathcal{T}_n = \emptyset$  then for every  $\sigma$  there is  $k < n$  that

$$(1.4) \quad \phi(\sigma_1 \sigma_2 \dots \sigma_k) < \Phi_*^k.$$

and hence

$$\phi(\sigma_1 \sigma_2 \dots \sigma_k)^{\frac{1}{k}} \leq \Phi_* - \varepsilon$$

since  $X_i$  is compact and  $\phi$ , restricted to  $X_i$ , is continuous.

Every subword of  $\sigma$  of length  $n$  contains index  $k$  with (1.4). Hence for every  $\sigma$  and  $m > n$

$$\phi(\sigma_1 \sigma_2 \dots \sigma_m) \leq \phi(\sigma_1 \sigma_2 \dots \sigma_{k_1}) \phi(\sigma_{k_1+1} \sigma_{k_1+2} \dots \sigma_{k_2}) \dots \phi(\sigma_{k_l+1} \sigma_{k_l+2} \dots \sigma_m)$$

where

$$\phi(\sigma_{k_j+1} \sigma_{k_j+2} \dots \sigma_{k_{j+1}}) < \Phi_*^{k_{j+1}-k_j} \quad j = 1, \dots, l.$$

It follows

$$\phi(\sigma_1 \sigma_2 \dots \sigma_n) \leq C(\Phi_* - \varepsilon)^{k_l}$$

where  $C = \max\{1, \Phi_1^n\}$ ,  $k_i < n$  and  $m - n < k_l < m$ .

For every  $m > n$  consider the maximal word  $\Phi_m$ . I.e let  $(\sigma_1^{[m]} \dots \sigma_m^{[m]})$  be the word satisfied (1.1). Then

$$(1.5) \quad \Phi_m^{\frac{1}{m}} \leq C^{\frac{1}{m}} (\Phi_* - \varepsilon)^{\frac{k_l}{m}}$$

and (1.5) is valid also if  $m \rightarrow \infty$ . Thus the contradiction  $\Phi_* \leq \Phi_* - \varepsilon$  is obtained and hence  $\mathcal{T}_n$  is not empty.  $\square$

*Remark 1.3.* An important case is when  $\{X_i \mid i = 1, 2, \dots\}$  are finite sets. In this case we can drop the continuity conditions, obviously.

That the theorem is not trivial also in this case, shows the next example.

Let

$$(1.6) \quad A_0 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix} \quad A_1 = \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}$$

and

$$\phi(\sigma_1 \sigma_2 \dots \sigma_n) = \|A_{\sigma_1} A_{\sigma_2} \dots A_{\sigma_n}\| \quad \sigma_i \in \{0, 1\}.$$

Then  $\phi$  is submultiplicative and normalized the unit matrix, we have

$$\left\| \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\| = a$$

and hence for the periodic sequence

$$(1.7) \quad \|A_0 A_1 \dots A_0 A_1\| = 30^{2n}.$$

It follows that (1.7) is  $\Phi_n$  for every  $n$  and  $\Phi_* = 30$ .

However, considered the set

$$(1.8) \quad \|A_0 A_1 \dots A_0 A_1 \dots A_0 \dots\| = 30^{2m} 0.3^{n-2m} \quad m = 1, 2, \dots$$

i.e. periodic product till  $2m$  ended by all  $A_0$ , we can find from this set  $\Phi_m$  for every even  $m$ , however each infinite product of (1.8) tends to zero.

**Theorem 1.4.** *If we restrict to a closed subset  $\mathcal{K}$  of  $\Pi_{i=1}^\infty X_i$  then our theorem remains valid if  $\mathcal{K}$  is shift-invariant in the sense*

$$\sigma_1 \sigma_2 \dots \sigma_n \dots \in \mathcal{K} \Rightarrow \sigma_2 \sigma_3 \dots \sigma_{n+1} \dots \in \mathcal{K}.$$

*Thus, as we shall see by the next examples, our theorem will be more applicable.*

*Proof.* The proof is the same as Theorem 1.2, with obvious modifications.  $\square$

*Remark 1.5.* The subset of  $\mathcal{K}$  with  $\phi_*(\sigma) = \Phi_*$  form a closed shift-invariant subspace.

**Theorem 1.6.** *If  $\sigma \in \mathcal{K}$  satisfies (1.3) then*

$$\Phi_n^{\frac{1}{n}} \geq \phi(\sigma_1, \sigma_2 \dots \sigma_n)^{\frac{1}{n}} \geq \Phi_* \quad n = 1, 2, \dots$$

and hence

$$(1.9) \quad \phi(\sigma_1, \sigma_2 \dots \sigma_n)^{\frac{1}{n}} \rightarrow \Phi_*.$$

*It is easy to verify that also (1.2)  $\Leftrightarrow$  (1.6) holds.*

*Proof.* The theorem follows from the submultiplicativity of sequence  $\{\Phi_n \mid n = 1, 2, \dots\}$ .  $\square$

## 2. EXAMPLES TOWARDS POSSIBLE APPLICATIONS

Each of the following example serves as a hint for possible application.

**Example 2.1.** Let  $\{X_i \mid i = 1, 2, \dots\}$  be bounded sets  $\{\mathcal{M}_i \mid i = 1, 2, \dots\}$  of square matrices with equal size, moreover,

$$\phi(\sigma_1 \sigma_2 \dots \sigma_k) = \|A_{\sigma_1} A_{\sigma_2} \dots A_{\sigma_k}\|$$

where  $A_{\sigma_i} \in \mathcal{M}_{\sigma_i}$ . I.e. a flow is represented by an infinite product of matrices.

In this setup we obtain the following

*Proposition.* The infinite product

$$A_{\sigma_1} A_{\sigma_2} \dots A_{\sigma_k} \dots \quad (A_{\sigma_k} \in \mathcal{M}_{\sigma_k})$$

tends to 0 for every  $\sigma$  if and only if there exists  $N$  that

$$(2.1) \quad \|A_{\sigma_1} A_{\sigma_2} \dots A_{\sigma_N}\| < 1$$

for every  $\sigma$ .

In this case

$$\Phi_* = \lim_{n \rightarrow \infty} \|A_{\sigma_1} A_{\sigma_2} \dots A_{\sigma_n}\|^{\frac{1}{n}}$$

called *the joint spectral radius* [1].

**Example 2.2.** The finite set  $\{F_1, F_2 \dots F_N\}$  of functions of a metric space  $X$  is called *joint contraction* if there is  $M$  such that every composition

$$(2.2) \quad F_{\sigma_1} \circ F_{\sigma_2} \circ \dots \circ F_{\sigma_M} \quad \sigma_i \in \{1, 2, \dots, N\}$$

of length  $M$  is a contraction.

*Proposition* Let  $\{F_1, F_2 \dots F_N\}$  be functions of a compact metric space  $X$  with Lipschitz constant

$$\sup \frac{d[F_i(x) - F_i(y)]}{d[x - y]} = r_i \quad i = 1, 2, \dots, N.$$

Then the sequence  $\{F_{\sigma_1} \circ F_{\sigma_2} \circ \dots \circ F_{\sigma_n}(x) \mid \sigma_i \in \{1, 2, \dots, N\}\}$  is convergent for every  $x \in X$  and the limit is independent of  $x$  if and only if  $\{F_1, F_2, \dots, F_N\}$  is joint contraction.

*Proof.* The "if" part can be proved in the same way as the Contraction Mapping Theorem. The "only if" part follows from Theorem 1.2 In fact, the Lipschitz constant is submultiplicative. If  $F$  and  $G$  are Lipschitz function with Lipschitz constant  $r_F$  and  $r_G$  then

$$\frac{d[F \circ G(x), F \circ G(y)]}{d[x, y]} \leq \frac{d[F \circ G(x), F \circ G(y)]}{d[G(x), G(y)]} \frac{d[G(x), G(y)]}{d[x, y]} \leq r_F r_G.$$

It follows, that if  $\phi(\sigma_1 \sigma_2 \dots \sigma_n)$  is the Lipschitz constant of  $F_{\sigma_1} \circ F_{\sigma_2} \circ \dots \circ F_{\sigma_n}$  then  $\phi$  is submultiplicative and hence there is a flow  $\sigma$  that

$$\sup_{x, y} \frac{d[F_{\sigma_1} \circ F_{\sigma_2} \circ \dots \circ F_{\sigma_n}(x), F_{\sigma_1} \circ F_{\sigma_2} \circ \dots \circ F_{\sigma_n}(y)]}{d[x, y]} \geq \Phi_*^n$$

from Theorem 1.2.

If  $\{F_1, F_2 \dots F_N\}$  is NOT joint contraction, then  $\Phi_n \geq 1$  for every  $n$  and hence  $\Phi_* \geq 1$ . It follows that there is a flow  $\sigma^*$  such that the Lipschitz constant of

$F_{\sigma_1^*} \circ F_{\sigma_2^*} \circ \cdots \circ F_{\sigma_n^*}$  is equal or greater then 1. I.e. for every  $n$  we have  $x'_n, y'_n \in X$  that

$$\frac{d[F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(x'_n), F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(y'_n)]}{d[x'_n, y'_n]} \geq \Phi_*^n.$$

Consider the sequences  $\{x'_n \mid n = 1, 2, \dots\}$  and  $\{y'_n \mid n = 1, 2, \dots\}$ . Since  $X$  is compact, we have convergent subsequences with limit  $x'$  and  $y'$  and after a straightforward calculation we obtain also

$$\frac{d[F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(x'), F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(y')]}{d[x', y']} \geq \Phi_*^n.$$

It follows that the convergence of

$$\{F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(x); \ n = 1, 2, \dots\}$$

depends on  $x$ , moreover, the sequence tend to infinity in an exponential rate  $\Phi_*^n$  if  $\Phi_* > 1$ .  $\square$

**Remark 2.3.** If  $\{F_1, F_2, \dots, F_N\}$  is NOT joint contraction, then often we can choose a subset  $\mathcal{K}$  that the sequences  $\{F_{\sigma_1} \circ F_{\sigma_2} \circ \cdots \circ F_{\sigma_n}(x) \mid n = 1, 2, \dots\}$ , corresponded to  $\mathcal{K}$ , are convergent with limit independent of  $x$ .

The set  $\mathcal{K}$  is obtained by to select words  $\sigma_i \sigma_{i+1} \dots \sigma_j$  and considered those sequences which do not contain  $F_{\sigma_i} \circ F_{\sigma_{i+1}} \circ \cdots \circ F_{\sigma_j}$ . Thus for the infinite compositions belonging to  $\mathcal{K}$  there is  $M$  that every composition (2.2) is a contraction.

**Example 2.4.** [3] Let  $\sigma$  be the input of a program  $P$  of a Turing machine and  $s_n(\sigma)$  be the number of *different* cells visited during  $n$  step of the execution of  $\sigma$ . We are interested in the input  $\sigma$  that takes the average

$$\frac{s_n(\sigma)}{n}$$

maximal after long time. There is subadditivity

$$s_{k+m}(\sigma) \leq s_k(\sigma) + s_m(P^k(\sigma))$$

where  $P^k(\sigma)$  is the output after the  $k$ -th step of the execution of  $\sigma$  ( $\sigma$  is a string with finite alphabet). Hence, applied Theorem 1.2 when  $\phi(\sigma_1 \sigma_2 \dots \sigma_n) = s_n(\sigma)$ , there exists input  $\sigma$  with

$$\frac{s_n(\sigma)}{n} \geq \Phi_* \quad n = 1, 2, \dots$$

and this is also the desired maximum.

**Example 2.5.** [5] Let  $(X, d)$  be a compact metric space and  $T : X \Rightarrow X$  a continuous map. For every open cover  $\mathcal{U}$  of  $X$  there is a finite subcover  $\mathcal{V}$ , since  $X$  is compact. Define  $N(\mathcal{U})$  as the minimal number of a subcover.

If  $\mathcal{U}, \mathcal{V}$  are open covers, then

$$\mathcal{U} \vee \mathcal{V} = \{u_i \cap v_j; \ u_i \in \mathcal{U}, v_j \in \mathcal{V}\}.$$

It easy to check that  $N(\bullet)$  is subadditive in the following sense

$$N(\mathcal{U} \vee \mathcal{V}) \leq N(\mathcal{U}) + N(\mathcal{V}).$$

The *topological entropy*  $h(T\mathcal{U})$  is defined by these concepts as follows

$$h(T, \mathcal{U}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1})$$

where

$$N(\mathcal{U}_0^{n-1}) = \bigvee_{k=1}^{n-1} T^{[-k]} \mathcal{U}.$$

The limit exists because of the subadditivity of  $N(\bullet)$  with  $a_n = \log N(\mathcal{U}_0^{n-1})$ .

The *topological entropy*  $h(T)$  of  $T$  is the supremum of  $\{h(T, \mathcal{U}) \mid \mathcal{U} \text{ is an open cover of } X\}$ .

If there is a compact topology of the minimal covers  $\mathcal{V}$  induced by the open sets of  $X$  such that

$$\mathcal{V} \implies N(\mathcal{U})$$

is continuous, then there is a cover  $\mathcal{U}$  of the compact space  $X$  that  $h(T) = h(T, \mathcal{U})$ .

#### REFERENCES

1. Daubechies, I. and Lagarias, J.C., *Sets of matrices all infinite products of which converge* Linear Algebra Appl. **161** (1992), 239.
2. Fekete, Michael, *Über der Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten* Mathematische Zeitschrift **17** (1923), 228-249.
3. Jeandel, Emmanuel, *Computability of the Entropy of one-tape Turing Machine* arXiv: 1302.1170 (2013)
4. Mate, László, *On infinite composition of affine mappings* Fundamenta Mathematicae **159** (1999) 85-90
5. Sarig, Omri, *Lecture Notes on Ergodic Theory* Weizman Institute, Israel (2009)

Institute of Mathematics, Budapest University of Technology and Economics  
*E-mail address:* mate@math.bme.hu